

# Supersymmetric Gödel-type Universe in four Dimensions

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## Abstract

We generalize the classification of all supersymmetric solutions of pure  $N = 2$ ,  $D = 4$  gauged supergravity to the case when external sources are included. It is shown that the source must be an electrically charged dust. We give a particular solution to the resulting equations, that describes a Gödel-type universe preserving one quarter of the supersymmetries.

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In a recent paper [1], we classified all supersymmetric solutions of pure gauged  $N = 2$  supergravity in four dimensions, generalizing the results of Tod [2] in the ungauged case. Unlike [2], we did not include external sources, and to remedy this will be the main scope of this letter. The motivation for including sources is twofold. First of all, it is desirable to have a more complete treatment that considers also this case. Second, external dust sources (as well as a negative cosmological constant) are necessary ingredients to obtain the Gödel universe [3] or its generalizations [4].

These spacetimes, which suffer from closed timelike curves (CTCs), have become fashionable recently, after the discovery of a maximally supersymmetric analogue of the Gödel universe in five dimensions [5]. Generalizations to ten and eleven dimensions have been found in [6, 7], and a variety of black holes embedded in Gödel spacetimes have been discussed in [8, 9, 10]. Various mechanisms of chronology protection using holographic screening [11] or supertubes [12] appeared in the literature. Particle and brane probes in these backgrounds were analyzed in [11, 12, 13]. Finally, string propagation on Gödel universes, which are T-dual to compactified pp-waves [11], was initiated in [14].

The study of these solutions hopefully will shed some light on holography for spaces more general than AdS, and on the resolution of CTCs by string theory. One might thus ask whether there are members of the family of Gödel-type solutions in four dimensions [4] that preserve some supersymmetry. As we said, these spacetimes have the common feature of a negative cosmological constant and external dust sources, so a natural framework to address this question is  $N = 2$ ,  $D = 4$  gauged supergravity with sources. In ungauged four-dimensional  $N = 2$  supergravity, supersymmetry imposes the condition that the sources form a perfect fluid with vanishing pressure [2]. If this holds true also in the gauged case, it is exactly what we want in order to obtain Gödel-type solutions. Indeed, it will be shown below that also in the gauged version, the source must be a charged dust; the only difference is that now the dust carries only electric charge, whereas the magnetic charge has to vanish.

In what follows, we use the conventions of [1]. In that paper, we obtained the general solution of  $N = 2$ ,  $D = 4$  gauged supergravity that admits at least one Killing spinor  $\epsilon$ . The solutions fall into two classes depending on whether the Killing vector  $V^\mu = i\bar{\epsilon}\Gamma^\mu\epsilon$  constructed from  $\epsilon$  is timelike or lightlike. To keep things short, we will generalize only the timelike case to the inclusion of external sources. The generalization of the lightlike case will be discussed elsewhere.

Let us briefly recall the results of [1] for timelike  $V^\mu$  (rewritten here in a

slightly more compact form). The general BPS solution reads<sup>1</sup>

$$\begin{aligned} ds^2 &= -\frac{4}{\ell^2 F \bar{F}}(dt + \omega_i dx^i)^2 + \frac{\ell^2 F \bar{F}}{4}[dz^2 + e^{2\phi}(dx^2 + dy^2)], \\ \mathcal{F} &= \frac{\ell^2}{4} F \bar{F} \left[ V \wedge df + * \left( V \wedge \left( dg + \frac{1}{\ell} dz \right) \right) \right], \end{aligned} \quad (1)$$

where  $i = 1, 2$ ;  $x^1 = x, x^2 = y$ , and we defined  $\ell F = 2i/(f - ig)$ , with  $f = \bar{\epsilon}\epsilon$  and  $g = i\bar{\epsilon}\Gamma_5\epsilon$ . Here,  $1/\ell$  is the minimal coupling between the graviphoton and the gravitini, which is related to the cosmological constant by  $\Lambda = -3\ell^{-2}$ . The timelike Killing vector is given by  $V = \partial_t$ . The functions  $\phi, F, \bar{F}$ , that depend on  $x, y, z$ , are determined by the system

$$\Delta F + e^{2\phi}[F^3 + 3FF' + F''] = 0, \quad (2)$$

$$\Delta\phi + \frac{1}{2}e^{2\phi}[F' + \bar{F}' + F^2 + \bar{F}^2 - F\bar{F}] = 0, \quad (3)$$

$$\partial_z\phi - \text{Re}(F) = 0, \quad (4)$$

where  $\Delta = \partial_x^2 + \partial_y^2$ , and a prime denotes differentiation with respect to  $z$ . (2) comes from the combined Maxwell equation and Bianchi identity, whereas (3) results from the integrability condition for the Killing spinor  $\epsilon$ . Finally, the shift vector  $\omega$  is obtained from<sup>2</sup>

$$\begin{aligned} \partial_z\omega_i &= \frac{\ell^4}{8}(F\bar{F})^2\epsilon_{ij}(f\partial_j g - g\partial_j f), \\ \partial_i\omega_j - \partial_j\omega_i &= \frac{\ell^4}{8}(F\bar{F})^2e^{2\phi}\epsilon_{ij}\left(f\partial_z g - g\partial_z f + \frac{2f}{\ell}\right), \end{aligned} \quad (5)$$

with  $\epsilon_{12} = 1$ .

If we allow external charged sources carrying electric current  $J_\mu^e$  and magnetic current  $J_\mu^m$ , the Maxwell equations read

$$\nabla^\mu \mathcal{F}_{\mu\nu} = -4\pi J_\nu^e, \quad \nabla_{[\mu} \mathcal{F}_{\nu\rho]} = \frac{4\pi}{3}\epsilon_{\mu\nu\rho}{}^\sigma J_\sigma^m. \quad (6)$$

Note that the inclusion of magnetic currents modifies the Bianchi identity, so that it is no more possible to define an electromagnetic vector potential  $\mathcal{A}_\mu$  in presence of continuous magnetic charge distributions. As  $\mathcal{A}_\mu$  appears explicitly in the supercovariant derivative

$$\mathcal{D}_\mu = \nabla_\mu - \frac{i}{\ell}\mathcal{A}_\mu + \frac{1}{2\ell}\Gamma_\mu + \frac{i}{4}\mathcal{F}_{\alpha\beta}\Gamma^{\alpha\beta}\Gamma_\mu \quad (7)$$

<sup>1</sup>We have chosen the conformal gauge for the two-metric  $h_{ij}$  appearing in [1].

<sup>2</sup>It can be shown that the integrability conditions for (5) follow from the Maxwell equations, even in presence of external sources.

of gauged supergravity, consistency requires setting the magnetic current  $J^m$  to zero. For the time being, we will keep  $J^m$ , and show at the end that conservation of the combined energy-momentum tensor of the electromagnetic field and the sources also leads to the condition of vanishing  $J^m$ . The charged sources carry some energy-momentum  $T_{\mu\nu}^{\text{ext}}$ . Imposing the integrability conditions, i.e. the vanishing of the supercurvature [1],

$$\begin{aligned} [\mathcal{D}_\nu, \mathcal{D}_\mu]\epsilon = & \left[ \frac{1}{\ell}(*\mathcal{F}_{\nu\mu}\Gamma_5 - i\mathcal{F}_{\nu\mu}) + \frac{1}{2\ell^2}\Gamma_{\nu\mu} + \frac{1}{4}\mathcal{R}^{ab}{}_{\nu\mu}\Gamma_{ab} \right. \\ & - \mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta[\nu}\Gamma_{\mu]\alpha} + \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\Gamma_{\nu\mu} - \frac{i}{\ell}\mathcal{F}^\alpha{}_{[\nu}\Gamma_{\mu]\alpha} \\ & \left. - \frac{i}{2}\Gamma_{\alpha\beta[\nu}\nabla_{\mu]}\mathcal{F}^{\alpha\beta} - i\nabla_{[\nu}\mathcal{F}_{\mu]}{}^\alpha\Gamma_\alpha \right] \epsilon = 0, \end{aligned} \quad (8)$$

we obtain, after some algebra, the BPS conditions on the currents,

$$J = J^e + iJ^m = \kappa(f - ig)V, \quad (9)$$

where  $\kappa$  is up to now an arbitrary real function. Conservation of  $J$  requires that  $\kappa$  be time-independent. The Einstein equations imply that these sources have the dust stress tensor

$$T_{\mu\nu}^{\text{ext}} = \kappa V_\mu V_\nu. \quad (10)$$

As the Einstein tensor is conserved, the same must be true for the combined energy-momentum tensor of the electromagnetic field and the sources. This leads to  $g = 0$  and thus, from (9), to  $J^m = 0$ <sup>3</sup>, which, as pointed out above, is already necessary in order to define a gauge potential  $\mathcal{A}$ . We set thus  $J^m = 0$  in what follows.

We can now follow closely the discussion of the sourceless timelike solutions in [1]. In fact, the only difference is in Maxwell's equations which include now the sources. The general supersymmetric solution with timelike Killing vector  $V^\mu$  is therefore still given by (1), (3) and (5), but with (2) replaced by

$$\Delta F + e^{2\phi}[F^3 + 3FF' + F'' + 4\pi\kappa F] = 0. \quad (11)$$

Furthermore, the condition  $g = 0$  yields  $\bar{F} = -F$ . The imaginary part of equation (3) implies then  $F' = 0$ , so that the functions  $F$  and  $\phi$  are determined by solving the system

$$\Delta F + e^{2\phi}[F^3 + 4\pi\kappa F] = 0, \quad (12)$$

$$\Delta\phi + \frac{3}{2}e^{2\phi}F^2 = 0. \quad (13)$$

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<sup>3</sup>The violation of the divergence of the stress tensor is proportional to  $\ell^{-1}$ , and thus vanishes in the ungauged case  $\ell \rightarrow \infty$ , where  $J^m = 0$  is no more necessary [2].

Let us assume that  $\kappa$  is a positive constant. Then a simple solution to (12) is given by  $F = i\sqrt{4\pi\kappa}$ . Equation (13) becomes the Liouville equation and describes a two-manifold  $h_{ij} = e^{2\phi}\delta_{ij}$  of constant curvature  $-12\pi\kappa$ . For convenience, we choose the parabolic Liouville solution to cast the transverse metric in Poincaré coordinates. This leads to the metric

$$ds^2 = -\frac{1}{\pi\kappa\ell^2} \left( dt - \frac{2\ell^2\sqrt{\pi\kappa}}{3x} dy \right)^2 + \pi\kappa\ell^2 dz^2 + \frac{\ell^2}{6x^2} (dx^2 + dy^2), \quad (14)$$

with uniform magnetic flux through the transverse manifold,

$$\mathcal{F} = \frac{\ell}{6x^2} dx \wedge dy. \quad (15)$$

A detailed analysis shows that this solution preserves exactly 1/4 of the original supersymmetry. The resulting spacetime has the Gödel-type metric with vorticity  $\Omega = 2/\ell$  and parameter  $m = \sqrt{6}/\ell$  given in [4], and represents a spacetime homogeneous universe with rigidly rotating dust and a magnetic field. The presence of CTCs is most obvious in cylindrical coordinates,

$$ds^2 = -\left( dt - \frac{4\Omega}{m^2} \sinh^2\left(\frac{mr}{2}\right) d\varphi \right)^2 + \frac{1}{m^2} \sinh^2(mr) d\varphi^2 + dr^2 + dz^2. \quad (16)$$

For sufficiently large  $r$  the vector  $\partial_\varphi$  is timelike, and its integral curves become CTCs.

In the ungauged case, the charged dust source can be interpreted as the dilaton field of the  $N = 4$  theory [15]. It would be interesting to see whether such an identification is possible also in the gauged case, for example by embedding the solution into  $N = 4$  gauged supergravity.

## Acknowledgements

This work was partially supported by INFN, MURST and by the European Commission RTN program HPRN-CT-2000-00131, in which M. M. C. and D. K. are associated to the University of Torino.

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